Dispersion relation and the dielectric tensor for magnetized plasmas with inhomogeneous magnetic field

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We investigate the dispersion relation for a magnetized plasma with weak magnetic field gradients perpendicular to the ambient magnetic field. An explicit expression for the effective dielectric tensor is derived, incorporating the relevant contributions due to the inhomogeneity, which include corrections to all orders in the small parameter ϵ, where $\epsilon = [(1/B_0)(dB_0/dx)]$. It is shown that this effective dielectric tensor satisfies the required symmetry conditions and is the tensor which should be utilized in the dispersion relation, in order to describe correctly wave-particle interactions in media with inhomogeneous magnetic field. The case of high frequency oscillations propagating perpendicularly to the magnetic field in a Maxwellian plasma is considered as an example and the effect of inhomogeneities in the magnetic field upon the absorption coefficient and the optical depth of ordinary mode waves is discussed. A region of negative absorption coefficient is predicted near the electron cyclotron frequency for sufficiently high inhomogeneity. Moreover, it is shown that significant differences may exist between the absorption coefficient evaluated with the present formulation and results from other approaches found in the literature which do not exhibit correct symmetry properties.

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I. INTRODUCTION

The subject of wave propagation in inhomogeneous plasmas is far from being simple from a mathematical point of view, since it involves Maxwell's equations for the electromagnetic field components, coupled to equations which describe the charge and current densities. In a collisionless plasma, these are a set of Vlasov equations for the distributions of each of the plasma species. However, in the case in which the wavelengths of the oscillations are much smaller than typical scale lengths of inhomogeneities, the treatment of the problem can be simplified. The fluctuations can be described by means of a WKB approximation and a local relation can be assumed between the current density and the electric field. The resulting dispersion relation is then valid at each point and its solution for a given wave frequency gives the local refraction index. This approach can be called the "locally homogeneous approximation" and is frequently employed in order to gain information about wave propagation and absorption in inhomogeneous plasmas.

A further step in the inclusion of inhomogeneity effects in the description of the dielectric effects is to take into account in the evaluation of the dielectric tensor components the space derivatives of the parameters that describe the plasma at each point, inserting them into the same dispersion relation previously mentioned [1]. The method has provided quite general expressions for the components of the dielectric tensor in the case of density and temperature inhomogeneities. This method has been applied to several situations, such as the study of low frequency waves and drift instabilities, with the wave vector perpendicular to the direction of the inhomogeneity [1–3], or the study of whistler instabilities due to inhomogeneous beams [4].

An important feature about these studies is related to the symmetry properties of the dielectric tensor. The dielectric tensor of a homogeneous plasma satisfies Onsager symmetry relations, but the addition of terms related to the derivatives of the plasma parameters destroys this symmetry, leading to the existence of nonresonant contributions to the anti-Hermitian parts of the dielectric tensor components [5]. This feature persists for any angle of propagation relative to the direction of the inhomogeneity, except for propagation perpendicular to the inhomogeneity. From another point of view, it has been demonstrated that this approach for the derivation of the dielectric tensor leads to a tensor depending on the wave frequency and the wave vector, which is not in fact the Fourier transform of the dielectric tensor in real space [6,5]. As a consequence, when this tensor is introduced into the dispersion relation, it does not describe adequately the dielectric properties and the exchange of energy between wave and particles.

A significant evolution of the treatment of the problem was achieved by the application of an iterative procedure to the wave equation, which has shown that the dielectric properties of the plasma may be well described by a dispersion relation that is formally the same as in a homogeneous plasma, but with an effective dielectric tensor replacing the homogeneous plasma dielectric tensor [6]. The effective dielectric tensor obtained according to the iterative procedure developed by Beskin, Gurevich, and Istomin in Ref. [6] (denoted in what follows as BGI) should be adequate for the description of the dielectric properties of the plasmas and to the correct description of the energy exchange between waves and particles, since it would be the actual Fourier transform of the dielectric tensor in real space [6]. In other words, the effective dielectric tensor is constructed in order to ensure that the absorption coefficient obtained from the solution of the dispersion relation is really connected to the energy exchange between a wave and particles [6,5,7].

The iterative procedure as proposed by BGI assumes that the gradients in the physical parameters are sufficiently weak such that a WKB approach is justified and mode conversion and reflection are ignored. This is a sensible assumption which may be violated in parameter regions where two dispersion curves approach each other, or in a resonance region, or when the wave approaches a "cutoff." The description of mode conversion phenomena in these localized regions is certainly an interesting theme of research and recent developments continue to appear in the literature [8]. However, many interesting wave phenomena occur in inhomogeneous plasmas, for which the validity of a WKB approach is justified. Such is the case, for instance, of high frequency oscillations propagating perpendicularly to the magnetic field near the electron cyclotron resonance. A wave-dynamical treatment of ordinary mode waves propagating perpendicularly across the resonance layer has been made and demonstrated that the validity of the WKB approximation can be easily shown for tokamak parameters [9]. Moreover, it has been demonstrated by means of wave absorption calculations that the WKB approximation results are good, even when the formal validity conditions are violated [9].

Explicit expressions for the components of the effective dielectric tensor using the BGI approach were provided later on, valid for electromagnetic waves of any frequency propagating in an arbitrary direction relative to the magnetic field and to the direction of inhomogeneity [5]. In the derivation of these expressions, it was assumed that any of the parameters appearing in the distribution function could be inhomogeneous, as density or temperature, but the magnetic field was supposed to be homogeneous. The expressions obtained were fully relativistic, included inhomogeneity effects up to order L^{-1}, where L is the scale length of spatial variations, and were valid as long as the Larmor radius of the particles was small relative to L. Although the direction of wave propagation could be arbitrary, the direction of the inhomogeneity was assumed to be perpendicular to the magnetic field. An investigation about inhomogeneity effects on electron cyclotron absorption by a Maxwellian plasma near the fundamental electron cyclotron frequency was then conducted with the use of this formalism [7].

In the present paper we discuss the derivation of the dielectric tensor for the case where the magnetic field is inhomogeneous. The case of the inhomogeneous magnetic field has a fundamental difference relative to the case of the homogeneous field because the unperturbed orbits of the plasma particles are affected by the magnetic field inhomogeneity, while they are not affected by inhomogeneities in other plasma parameters such as density and temperature. As a consequence, the resonance condition in momentum space is affected by the inhomogeneity and an infinite number of corrections is necessary to be added in order to build up the effective dielectric tensor [6]. We also discuss the symmetry properties of the effective dielectric tensor and compare with other approaches found in the literature for the description of dielectric properties of plasmas in inhomogeneous magnetic fields.

The plan of the paper is the following. In Sec. II we describe the physical system to be considered and the evaluation of the unperturbed orbits of the plasma particles in an inhomogeneous field. In Sec. III we give a short account of the procedures to be employed in the derivation of the effective dielectric tensor for electromagnetic waves in an inhomogeneous magnetized plasma. Section IV is reserved for a discussion about the symmetry properties of the effective dielectric tensor and for a comparison with other approaches utilized in the literature. In Sec. V we apply our expressions for the case of high frequency waves propagating perpendicularly to the magnetic field, for a particular distribution function, and show the effect of the inhomogeneity of the magnetic field on the absorption coefficient and the optical depth of ordinary mode waves propagating in an inhomogeneous slab of plasma with parameters in the range of the parameters of present day tokamaks. Finally, Sec. VI is reserved for conclusions.

II. PARTICLE ORBITS IN INHOMOGENEOUS MAGNETIC FIELDS

In what follows we will consider the magnetic field pointing in the z direction and inhomogeneous in the x direction $\mathbf{B}_0 = B_0(x)\mathbf{e}_z$. The waves will be assumed to be propagating in arbitrary directions, making an angle θ with the direction of the magnetic field and an angle ψ between the perpendicular wave vector and the direction of the inhomogeneity. In the proposed geometry, the relevant constants of motion are the quantities

$$p_\perp^2, p_\parallel, \overline{P}_\alpha \equiv p_\perp \sin\varphi + m_\alpha \int^x \Omega_\alpha(x')dx' \ , \qquad (1)$$

where $p_\perp$ and $p_\parallel$ are the components of the particle momentum respectively perpendicular and parallel to the magnetic field, $\overline{P}_\alpha/(m_\alpha\Omega_\alpha)$ is the x coordinate of the guiding center position, x is the coordinate of the particle position along the x axis, φ is the angle between the vector $\mathbf{p}_\perp$ and the x direction, m_α is the particle rest mass, and $\Omega_\alpha(x) = q_\alpha B_0(x)/m_\alpha c$ is the particle cyclotron angular frequency; q_α is the particle charge and c is the velocity of light.

The constants of motion are important in the present context because in a collisionless plasma, the equilibrium distribution function for a plasma species α satisfies the Vlasov equation and is an arbitrary function of the constants of motion. The unperturbed orbits of particles of species α are the solutions of the equations

$$\frac{d\mathbf{x}'_\alpha}{dt'} = \frac{\mathbf{p}'_\alpha}{m_\alpha\gamma'_\alpha} \ ,$$
$$\frac{d\mathbf{p}'_\alpha}{dt'} = \frac{\Omega_\alpha(x'_\alpha)}{\gamma'_\alpha}\mathbf{p}'_\alpha \times \mathbf{e}_z \ , \qquad (2)$$

with the initial conditions given by

$$\mathbf{x}'_\alpha(t'=t)=\mathbf{x},$$
$$\mathbf{p}'_\alpha(t'=t)=\mathbf{p}=\mathbf{e}_x p_\perp\cos\varphi+\mathbf{e}_y p_\perp\sin\varphi+\mathbf{e}_z p_\parallel\ . \quad (3)$$

In the case of weak inhomogeneities, the magnetic field can be represented by two terms of a Taylor series and the cyclotron frequency can be written as

$$\Omega_\alpha(x'_\alpha)\sim\Omega_\alpha(x)[1+\epsilon(x'_\alpha-x)], \quad (4)$$

where $\epsilon\equiv\left[\frac{1}{B_0}\frac{dB_0}{dx'_\alpha}\right]_{x'_\alpha=x}$. When this approximation is utilized in Eq. (2) and the orbit equations are correctly written up to order ϵ, care must be taken in order to avoid secular terms, which requires a nonlinear correction to the oscillation frequency. As a result, the orbit equations are written as a series in powers of ϵ, in which the coefficients incorporate an infinite number of corrections, through the nonlinear modification to the frequency, which appear as argument of trigonometric functions [10]. The correct orbits for particles of species α, correct up to order ϵ, are therefore

$$\begin{aligned}
p'_{x\alpha}(t') &= p_\perp\cos\theta_\alpha \\
&\quad +\epsilon\frac{s_\alpha}{m_\alpha\gamma_\alpha\beta_\alpha}p_\perp^2[-\cos\varphi\sin\theta_\alpha+\tfrac{1}{2}\sin 2\theta_\alpha]\ , \\
p'_{y\alpha}(t') &= -p_\perp\sin\theta_\alpha \\
&\quad +\epsilon\frac{s_\alpha}{m_\alpha\gamma_\alpha\beta_\alpha}p_\perp^2[\tfrac{1}{2}-\cos\varphi\cos\theta_\alpha+\tfrac{1}{2}\cos 2\theta_\alpha]\ , \\
p'_{z\alpha}(t') &= p_\parallel\ , \\
x'_\alpha(t') &= x+\frac{s_\alpha}{m_\alpha\gamma_\alpha\beta_\alpha}p_\perp(\sin\theta_\alpha+\sin\varphi) \\
&\quad +\epsilon\frac{p_\perp^2}{m_\alpha^2\gamma_\alpha^2\beta_\alpha^2}[\cos\varphi\cos\theta_\alpha-\tfrac{1}{4}\cos 2\theta_\alpha \\
&\quad -\tfrac{1}{4}(3\cos^2\varphi+\sin^2\varphi)]\ , \\
y'_\alpha(t') &= y+\frac{s_\alpha}{m_\alpha\gamma_\alpha\beta_\alpha}p_\perp(\cos\theta_\alpha-\cos\varphi) \\
&\quad +\epsilon\frac{p_\perp^2}{m_\alpha^2\gamma_\alpha^2\beta_\alpha^2}[\tfrac{1}{2}s_\alpha\beta_\alpha\tau-\cos\varphi(\sin\theta_\alpha+\sin\varphi) \\
&\quad +\tfrac{1}{4}(\sin 2\theta_\alpha+\sin 2\varphi)]\ , \\
z'_\alpha(t') &= z+\frac{p_\parallel}{m_\alpha\gamma_\alpha}\tau\ ,
\end{aligned} \quad (5)$$

where

$$\beta_\alpha\equiv\frac{|\,\hat\Omega_\alpha\,|}{\gamma_\alpha},\ \hat\Omega_\alpha\equiv\Omega_\alpha(x)+\epsilon\frac{p_\perp\sin\varphi}{m_\alpha},$$
$$s_\alpha\equiv\mathrm{sgn}(\Omega_\alpha),\ \theta_\alpha\equiv s_\alpha\beta_\alpha\tau-\varphi,\ \tau\equiv t'-t. \quad (6)$$

The orbits given by Eqs. (5) are consistent, in the sense that the momenta are obtained from the temporal derivatives of the position, and the initial conditions are satisfied. If $\epsilon=0$, the orbits given by Eqs. (5) reduce to the traditional expressions obtained in the homogeneous case. There is a secular term in the y direction, which is related to the macroscopic drift of the particles in the inhomogeneous field and does not disappear with the nonlinear correction to the frequency.

III. THE DIELECTRIC TENSOR FOR INHOMOGENEOUS PLASMAS

The procedure to be employed in order to obtain the effective dielectric tensor requires as an initial step the evaluation of a tensor in whose derivation a plane wave approximation is employed, as in the homogeneous case [6,5]. The Vlasov-Maxwell system is linearized and the method of the characteristics is utilized. Following the usual steps, it is possible to write the formal expression

$$\varepsilon^0_{ij}=\delta_{ij}-i\sum_\alpha X_\alpha\frac{\omega}{n_\alpha}\int d^3p\ \frac{p_iA_{\alpha j}}{\gamma_\alpha}\ , \quad (7)$$

where

$$X_\alpha=\frac{\omega_{p\alpha}^2}{\omega^2}=\frac{4\pi n_\alpha q_\alpha^2}{m_\alpha\omega^2}\ ,$$
$$\mathbf{A}_\alpha=\int_{-\infty}^0 d\tau\boldsymbol{\Theta}_\alpha\,e^{i[\mathbf{k}\cdot(\mathbf{x}'_\alpha-\mathbf{x})-\omega\tau]}\ ,$$
$$\boldsymbol{\Theta}_\alpha=\left(1-\frac{\mathbf{p}'\cdot\mathbf{k}}{m_\alpha\gamma_\alpha\omega}\right)\nabla_{\mathbf{p}'}F_{\alpha 0}+\frac{\mathbf{k}\cdot\nabla_{\mathbf{p}'}F_{\alpha 0}}{m_\alpha\gamma_\alpha\omega}\mathbf{p}'\ .$$

In these expressions, ω is the wave angular frequency and n_α is the density of particles of the species α.

The orbits of the particles have to be inserted in Eq. (7) in order that an explicit expression for the components of the tensor can be obtained. If we take Eqs. (5) with $\epsilon=0$, the result is the well-known dielectric tensor for a homogeneous medium. As the next step, instead of taking the orbits correct up to order ϵ, as given by (5), we consider only the most important contributions of the inhomogeneity to the orbits, namely, the macroscopic drift and the nonlinear correction to the frequency, which is essential in order to avoid undesirable secularities [11,12]. Moreover, we restrict the discussion to the case of high frequency oscillations, in which the ions can be considered as a neutralizing background and only the electrons need to be taken into account in the dispersion relation. This simplifies the notation, although it does not mean any loss of generality. In what follows we will avoid unnecessary use of the index e appended to the electronic quantities. We also choose the coordinate system in order to write our expressions around the position $x=0$ and therefore have the following orbit equations for the plasma electrons:

$$\begin{aligned}
p'_x &= p_\perp\cos\theta_e\ , \\
p'_y &= -p_\perp\sin\theta_e\ , \\
p'_z &= p_\parallel\ , \\
x'-x &= \frac{p_\perp}{m\Omega}(\sin\theta_e+\sin\varphi)\ , \\
y'-y &= \frac{p_\perp}{m\Omega}(\cos\theta_e-\cos\varphi)+\epsilon\frac{p_\perp^2}{2m^2\Omega\gamma}\tau\ , \\
z'-z &= \frac{p_\parallel}{m\gamma}\tau\ ,
\end{aligned} \quad (8)$$

where

$$\theta_e = \left[\Omega(x) + \epsilon\frac{p_y}{m}\right]\frac{\tau}{\gamma} - \varphi \ ,$$
$$\theta = \frac{\Omega}{\gamma}\tau - \varphi = \theta_e(\epsilon = 0) \ ,$$
$$\Omega(x) = \Omega(1+\epsilon x) \ ,$$
$$\Omega = \Omega_e = \frac{q_e B}{mc} \ .$$

In these expressions, m is the electron mass.

We now concentrate our efforts on the less general case of an electron distribution function $F_{e0} \equiv F_0(p_\perp^2, p_\parallel)$ since we wish to stress the study on inhomogeneities in the magnetic field; other kinds of inhomogeneities, where the distribution is explicitly dependent on position and therefore must exhibit the dependence on the constant of motion $\overline{P}_e$, were already discussed in our previous studies [5,7]. The ion distribution therefore carries the current in the y direction, which must exist in the plasma in order to satisfy the equilibrium configuration with the x-dependent magnetic field pointing in the z direction.

Therefore, with the electron distribution function $F_{e0} \equiv F_0(p_\perp^2, p_\parallel)$, the quantity Θ, which appear in (7), may be written as

$$\Theta = \frac{1}{p_\perp}\left\{p'_x\left[\mathbf{e}_x\Phi_0(F_0) + \mathbf{e}_z\frac{k_\perp}{m\gamma\omega}\mathcal{L}(F_0)\cos\psi\right] + p'_y\left[\mathbf{e}_y\Phi_0(F_0) + \mathbf{e}_z\frac{k_\perp}{m\gamma\omega}\mathcal{L}(F_0)\sin\psi\right] + \mathbf{e}_z p_\perp\partial_{p_\parallel}F_0\right\}, \quad (9)$$

where

$$\mathcal{L}(F_0) \equiv p_\parallel\partial_{p_\perp}F_0 - p_\perp\partial_{p_\parallel}F_0 \ ,$$
$$\Phi_0(F_0) \equiv \partial_{p_\perp}F_0 - \frac{k_\parallel}{m\gamma\omega}\mathcal{L}(F_0) \ ,$$
$$k_x = k_\perp\cos\psi \ ,$$
$$k_y = k_\perp\sin\psi \ ,$$
$$k_z = k_\parallel \ .$$

Moreover, with the unperturbed orbits given by Eqs. (8), we have

$$\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = b\sin(\theta_e+\psi) + b\sin(\varphi-\psi) + \left(\frac{k_\parallel p_\parallel}{m\gamma} + \frac{\epsilon b p_\perp\sin\psi}{2m\gamma}\right)\tau \ , \quad (10)$$

where $b \equiv k_\perp p_\perp/(m\Omega)$. The unperturbed orbits also appear in expression (9), with the result that we have in the integrand combinations of trigonometric functions and exponentials, which can be expanded as

$$e^{ib\sin(\theta_e+\psi)} = \sum_{n=-\infty}^{\infty} J_n(b)\, e^{in(\theta_e+\psi)} \ ,$$

$$\cos(\theta_e+\psi)\, e^{ib\sin(\theta_e+\psi)} = \sum_{n=-\infty}^{\infty}\frac{n}{b}J_n(b)\, e^{in(\theta_e+\psi)} \ , \quad (11)$$

$$\sin(\theta_e+\psi)\, e^{ib\sin(\theta_e+\psi)} = -i\sum_{n=-\infty}^{\infty}J'_n(b)\, e^{in(\theta_e+\psi)} \ .$$

With the use of these expansions, the τ integral can be easily performed and the quantity A_j can be written as

$$A_j = i\gamma\, e^{ib\sin(\varphi-\psi)}\sum_{n=-\infty}^{\infty}\frac{e^{-in(\varphi-\psi)}}{D_n} \times\left[\Phi_0(F_0)\beta_j^n - \delta_{j3}\frac{D_n}{\gamma\omega p_\perp}\mathcal{L}(F_0)J_n(b)\right] \ , \quad (12)$$

where

$$D_n = \omega\gamma - \frac{k_\parallel p_\parallel}{m} - \frac{\epsilon b p_\perp\sin\psi}{2m} - n\Omega(x) - \frac{n\epsilon p_y}{m} \ ,$$
$$\beta_1^n = \cos\psi\,\frac{n}{b}\,J_n(b) - i\,\sin\psi J'_n(b) \ ,$$
$$\beta_2^n = \sin\psi\,\frac{n}{b}\,J_n(b) + i\,\cos\psi J'_n(b) \ ,$$
$$\beta_3^n = \frac{p_\parallel}{p_\perp}J_n(b) \ .$$

We are then left with the momentum integrals to be performed. However, there is a practical difficulty. In the case of the homogeneous magnetic field, the quantity D_n, which appears in the denominator of the integrand, is independent of φ. In the case of azimuthally symmetric distribution functions, the φ integral is then trivial and the resonance condition $D_n = 0$ is satisfied over an ellipse in momentum space, the so-called "resonance ellipse" (actually a semiellipse since $p_\perp$ must be positive). However, in the case of an inhomogeneous magnetic field, the resonance condition is modified and becomes φ dependent. The φ dependence in the resonant denominator precludes an easy analytical solution of the momentum integrals which are necessary for the components of the dielectric tensor.

At this point, we adopt another approach. By the use of the identity

$$\frac{i}{D_n} \equiv \int_0^\infty d\tau\, e^{iD_n\tau} \ , \quad (13)$$

we reintroduce the τ dependence in the quantity A_j,

$$A_j = \gamma\, e^{ib\sin(\varphi-\psi)}\sum_{n=-\infty}^{\infty}\int_0^\infty d\tau\, e^{iD_n\tau} \times e^{-in(\varphi-\psi)}\Phi_0(F_0)\beta_j^n - \delta_{j3}\, i\, e^{ib\sin(\varphi-\psi)}\sum_{n=-\infty}^{\infty}\frac{e^{-in(\varphi-\psi)}}{\omega p_\perp} \times\mathcal{L}(F_0)J_n(b) \ . \quad (14)$$

Therefore, the ε_{ij}^0 components can be written as

$$\varepsilon_{ij}^0 = \delta_{ij} - \frac{X}{n_e}\delta_{j3}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int d^3p\,\frac{\mathcal{L}(F_0)}{\gamma p_\perp}J_n(b)P_i^n(0) - i\,\frac{X}{2\pi}\frac{\omega}{n_e}\sum_{n=-\infty}^{\infty}\int_0^\infty d\tau\int d^3p\;\Phi_0(F_0) \times\exp\left[i\left(\omega\gamma - \frac{k_\parallel p_\parallel}{m} - \frac{\epsilon b p_\perp}{2m}\sin\psi - n\Omega(x)\right)\tau\right] \times\beta_j^n P_i^n(\epsilon) \ , \quad (15)$$

where

$$P_i^n(\epsilon) = \int_0^{2\pi} d\varphi\, p_i\, e^{ib\sin(\varphi-\psi)} e^{-in(\varphi-\psi)} \times e^{-i\alpha_n\tau\sin\varphi} ,$$
$$\alpha_n = \frac{n\epsilon p_\perp}{m} .$$

We now define the quantities δ_n and θ_n,

$$\delta_n^2 = (b - \alpha_n\tau\cos\psi)^2 + (\alpha_n\tau\sin\psi)^2 ,$$
$$\theta_n = \tan^{-1}\left[\frac{\alpha_n\tau\sin\psi}{b - \alpha_n\tau\cos\psi}\right] \tag{16}$$

and write the quantity P_i^n as a function of these new variables

$$P_i^n(\epsilon) = e^{-in\theta_n} \int_0^{2\pi} d\varphi\, p_i\, e^{i\delta_n\sin(\varphi-\psi-\theta_n)} \times e^{-in(\varphi-\psi-\theta_n)} . \tag{17}$$

With the use of Bessel function expansions such as those in (11) for the p_i components, we arrive at

$$P_i^n(\epsilon) = 2\pi p_\perp\, e^{-in\theta_n} \chi_i^n , \tag{18}$$

where

$$\chi_1^n = \cos(\psi+\theta_n)\frac{n}{\delta_n}J_n(\delta_n) + i\,\sin(\psi+\theta_n)J_n'(\delta_n) ,$$
$$\chi_2^n = -i\,\cos(\psi+\theta_n)J_n'(\delta_n) + \sin(\psi+\theta_n)\frac{n}{\delta_n}J_n(\delta_n) ,$$
$$\chi_3^n = \frac{p_\|}{p_\perp}J_n(\delta_n) .$$

With the use of these expressions in (15), the ε_{ij}^0 components can be given by the expression

$$\varepsilon_{ij}^0 = \delta_{ij} - \frac{X}{n_e}\delta_{i3}\delta_{j3}\int d^3p\,\frac{\mathcal{L}(F_0)}{\gamma}\frac{p_\|}{p_\perp} + \chi_{ij}^0 , \tag{19}$$

where

$$\chi_{ij}^0 = -\,i\,X\frac{\omega}{n_e}\sum_{n=-\infty}^{\infty}\int_0^\infty d\tau\int d^3p\; p_\perp\Phi_0(F_0) \times\exp\left[i\left(\omega\gamma - \frac{k_\| p_\|}{m} - \frac{\epsilon b p_\perp}{2m}\sin\psi - n\Omega(x)\right)\tau\right] \times e^{-i\,n\theta_n}\beta_j^n\chi_i^n .$$

This tensor does not satisfy Onsager symmetry conditions for $\epsilon \neq 0$, as can be easily verified. This is an indication that it does not describe adequately the energy exchange between a wave and particles and therefore it is not the appropriate tensor to be used in the dispersion relation.

Another important point which must be noted is that the tensor $\overleftrightarrow{\varepsilon}^0$ is not really the Fourier transform of the dielectric tensor, as it would be in the case of a homogeneous medium. A general nonlocal linear relationship between the current density and the electric field may be written as $\vec{J}(\vec{r},t) = \int d\vec{r}'\,dt'\overleftrightarrow{\sigma}(\vec{r},\vec{r}',t,t')\cdot\vec{E}(\vec{r}',t')$. The Fourier transforms of $\vec{J}$ and $\vec{E}$ may be related as $\vec{J}(\vec{k},\omega) = \overleftrightarrow{\sigma}(\vec{k},\omega)\cdot\vec{E}(\vec{k},\omega)$ if one uses the convolution theorem with the hypothesis that the medium is homogeneous and stationary. In order that $\overleftrightarrow{\varepsilon}^0$ can be taken as the actual dielectric tensor in the dispersion relation, a homogeneity hypothesis must be assumed, something which is inconsistent with the keeping of terms due to the inhomogeneity in $\overleftrightarrow{\varepsilon}^0$.

If these fundamental features are disregarded and the $\overleftrightarrow{\varepsilon}^0$ tensor is introduced in the dispersion relation, the ensuing absorption coefficient displays terms that are not connected to the energy exchange between waves and particles. This is a common feature between the $\overleftrightarrow{\varepsilon}^0$ and other versions of the dielectric tensor obtained with similar approximations (i.e., which incorporate inhomogeneity effects, but are not really the Fourier transform of the space- and time-dependent dielectric tensor). The inadequacy of the dielectric tensor reveals itself in the lack of proper symmetry, which has a consequence that the anti-Hermitian parts contain nonresonant terms [5]. These nonresonant terms describe the variation of the wave amplitude due to the variation of the group velocity in an inhomogeneous medium, not true absorption or amplification [13,14]. The effective dielectric tensor obtained by the iterative procedure devised by BGI, on the other hand, features the anti-Hermitian part free of these nonresonant terms. When it is utilized in the dispersion relation, the absorption coefficient really describes absorption and/or amplification due to the wave-particle interaction [6,5].

The effective dielectric tensor, which should be used in the dispersion relation for the case of inhomogeneous plasmas, can be obtained by the addition of corrections to the tensor given by Eq. (19) [6,5]. In a previous study, for the case of a plasma with density and temperature inhomogeneities and homogeneous magnetic field, in the derivation of the equivalent to the present ε_{ij}^0 we have kept first-order corrections due to the inhomogeneity in the expansion of the space-dependent distribution function [5]. In such a case, it was sufficient to add first-order corrections to obtain the correct effective tensor. However, in the present case of an inhomogeneous magnetic field, the resonance condition itself is affected by the inhomogeneity and the addition of first-order corrections is not sufficient to arrive at the effective tensor with the correct properties [6]. The addition of corrections to all orders of the inhomogeneity parameter ϵ to the ε_{ij}^0 components is necessary. The task can be accomplished by the use of Eq. (21) of Ref. [6], which gives the components of the effective dielectric tensor as

$$\varepsilon_{ij}(\mathbf{x},\mathbf{k},\omega) = \frac{1}{(2\pi)^3}\int d^3k'\int d^3\eta\,\varepsilon_{ij}^0\left(\mathbf{x}+\frac{\boldsymbol{\eta}}{2},\mathbf{k}',\omega\right) \times e^{i(\mathbf{k}'-\mathbf{k})\cdot\boldsymbol{\eta}} . \tag{20}$$

After performing the integrations indicated in (20), for the situation which we are considering, we obtain

$$\varepsilon_{ij}(x,\mathbf{k},\omega) = \varepsilon_{ij}^0(x, k_x' = k_x + \epsilon_n, k_y, k_\|, \omega) , \tag{21}$$

where $\epsilon_n \equiv n\Omega\epsilon\tau/2$. It is convenient to observe here that $\alpha_n\tau = 2\epsilon_n b/k_\perp$. Defining now the quantity

$$\zeta_n \equiv \left[\left(\frac{\epsilon_n}{k_\perp} + \cos\psi\right)^2 + \sin^2\psi\right]^{1/2}, \tag{22}$$

it is possible to show that

$$\begin{aligned} k'_\perp &= k_\perp \zeta_n , \\ b' &= b\zeta_n , \\ \delta'_n &= b\zeta_{-n} , \\ \tan\theta'_n &= \frac{2(\epsilon_n/k_\perp)\sin\psi}{1-(\epsilon_n/k_\perp)^2} , \\ \beta_i'^n &= \pi_i^n , \\ \chi_i'^n &= (-1)^{n+1-\delta_{i3}}\pi_i^{-n} , \end{aligned}$$

where

$$\begin{aligned} \pi_1^n &= \frac{1}{\zeta_n}\left[\left(\cos\psi + \frac{\epsilon_n}{k_\perp}\right)\frac{n}{b\zeta_n} J_n(b\zeta_n) - i\sin\psi J'_n(b\zeta_n)\right] , \\ \pi_2^n &= \frac{1}{\zeta_n}\left[\sin\psi \frac{n}{b\zeta_n} J_n(b\zeta_n) + i\left(\cos\psi + \frac{\epsilon_n}{k_\perp}\right) J'_n(b\zeta_n)\right] , \\ \pi_3^n &= \frac{p_\parallel}{p_\perp} J_n(b\zeta_n), \end{aligned}$$

and the primed quantities denote those quantities appearing in ε_{ij}^0 that are dependent upon k'_x. Finally, we may then write the components of the effective dielectric tensor as

$$\varepsilon_{ij} = \delta_{ij} - \frac{X}{n_e}\delta_{i3}\delta_{j3}\int d^3p\, \frac{\mathcal{L}(F_0)}{\gamma}\frac{p_\parallel}{p_\perp} + \chi_{ij} , \tag{23}$$

where

$$\begin{aligned} \chi_{ij} = &-iX\frac{\omega}{n_e}\sum_{n=-\infty}^{\infty}\int_0^\infty d\tau \int d^3p\, p_\perp \Phi_0(F_0) \\ &\times \exp\left[i\left(\omega\gamma - \frac{k_\parallel p_\parallel}{m} - \frac{\epsilon b p_\perp}{2m}\sin\psi - n\Omega(x)\right)\tau\right] \\ &\times e^{-in\theta'_n}(-1)^{n+1-\delta_{i3}}\pi_i^{-n}\pi_j^n . \end{aligned} \tag{24}$$

This expression for the components of the effective dielectric tensor is one of the theoretical cornerstones of the present investigation. A simplified form, valid for propagation along the direction of the inhomogeneity, has already appeared in the literature [15].

IV. SYMMETRY PROPERTIES OF THE EFFECTIVE DIELECTRIC TENSOR AND COMPARISON WITH OTHER APPROACHES FROM THE LITERATURE

The effective dielectric tensor must satisfy the correct symmetry relations, the so-called Onsager relations, since it was built in order to describe correctly the energy exchange in the wave-particle interaction. As is well known, the Onsager relations are based upon the time reversal invariance of the microscopical equations of motion. In a time reversed situation, all the velocities are reversed and the symmetry is ensured by the change of $-\mathbf{B}_0$ for $\mathbf{B}_0$. The wave vector $\mathbf{k}$ also must be reversed [16]. Moreover, since the velocities of all particles are reversed, the distribution function of the particles must be reversed with respect to the velocity component parallel to the magnetic field. As a consequence, the symmetry condition that must be satisfied may be written as

$$\varepsilon_{ij}(\mathbf{B}_0, \mathbf{k}; F_0(p_\perp, p_\parallel)) = \varepsilon_{ji}(-\mathbf{B}_0, -\mathbf{k}; F_0(p_\perp, -p_\parallel)) . \tag{25}$$

The effective dielectric tensor given by Eq. (23) indeed satisfies this symmetry condition, which of course is the same condition satisfied by the dielectric tensor of a homogeneous plasma. For the homogeneous case, however, the symmetry condition can be cast in a simplified and more familiar form, namely, $\varepsilon_{ij}(\mathbf{B}_0) = \varepsilon_{ji}(-\mathbf{B}_0)$. This form is possible due to the peculiar form of the dielectric tensor in the homogeneous case and does not exclude or contradict the correct condition given by Eq. (25).

We now proceed by comparing the effective dielectric tensor obtained in the present paper with other forms of the dielectric tensor, utilized in the recent literature. They are derived under different assumptions, with the common feature that they are intended to be used in the usual dispersion relation, without the addition of any corrections such as those proposed by BGI.

A. Unperturbed orbits with drift term and without nonlinear correction to the frequency

Let us first discuss an apparently sound approximation, which considers that the unperturbed orbits are equal to the orbits in homogeneous fields, with the addition of terms due to the macroscopical drift [3]:

$$\begin{aligned} p'_x &= p_\perp \cos\theta , \\ p'_y &= -p_\perp \sin\theta + \epsilon\frac{p_\perp^2}{2m\Omega} , \\ p'_z &= p_\parallel , \\ x' - x &= \frac{p_\perp}{m\Omega}(\sin\theta + \sin\varphi) , \\ y' - y &= \frac{p_\perp}{m\Omega}(\cos\theta - \cos\varphi) + \epsilon\frac{p_\perp^2}{2m^2\Omega\gamma}\tau , \\ z' - z &= \frac{p_\parallel}{m\gamma}\tau , \end{aligned} \tag{26}$$

where we have used the same definitions as those utilized in (8). The formulation that utilizes these approximated orbits and can be found in Ref. [3] will be called in what follows approach A.

Following now the same steps as those utilized after Eq. (8), but with the approximated orbits (26), we arrive at an expression for the quantity A_j, which is similar to that of Eq. (12), with an added term, and with a different resonant denominator,

$$\begin{aligned} A_j = i\gamma e^{ib\sin(\varphi-\psi)} &\sum_{n=-\infty}^{\infty}\frac{e^{-in(\varphi-\psi)}}{\hat{D}_n} \\ &\times\left[\Phi_0(F_0)\beta_j^n - \delta_{j3}\frac{D_n}{\gamma\omega p_\perp}\mathcal{L}(F_0)J_n(b)\right. \\ &\left.+\delta_{j2}\frac{\epsilon p_\perp}{2m^2\Omega\gamma}\Phi_0(F_0)J_n(b)\right] , \end{aligned} \tag{27}$$

where

$$\hat{D}_n = \omega\gamma - \frac{k_\parallel p_\parallel}{m} - \frac{\epsilon b p_\perp \sin\psi}{2m} - n\Omega \ .$$

This expression has to be inserted into Eq. (7) in order to arrive at the dielectric tensor components. The φ integral is now readily performed since there is no φ dependence in the resonant denominator. For azimuthally symmetric distribution functions, the components of the dielectric tensor for high frequency oscillations can therefore be written as

$$\varepsilon^0_{ij} = \delta_{ij} + \frac{X\omega}{n_e}\sum_{n=-\infty}^{+\infty}\int d^3p\ p_\perp \frac{\varphi_0(F_0)}{\hat{D}_n}\left(\frac{p_\parallel}{p_\perp}\right)^{\delta_{iz}+\delta_{jz}} \times\left[R^R_{ij} + iR^I_{ij} + \delta_{jy}\frac{\epsilon p_\perp}{2m\Omega}(R^R_{iz} + iR^I_{iz})\right] - \delta_{iz}\delta_{jz}\frac{X}{n_\alpha}\int d^3p\ \frac{\mathcal{L}(F_0)}{\gamma}\left(\frac{p_\parallel}{p_\perp}\right) \ . \tag{28}$$

The quantities R_{ij} are given by

$$\begin{aligned}
R^R_{xx} &= J'^2_n(b) + \cos^2\psi\left(\frac{n^2}{b^2}J^2_n(b) - J'^2_n(b)\right) \ ,\\
R^I_{xx} &= 0 \ ,\\
R^R_{xy} &= \left(\frac{n^2}{b^2}J^2_n(b) - J'^2_n(b)\right)\sin\psi\cos\psi \ ,\\
R^I_{xy} &= \frac{n}{b}J_n(b)J'_n(b) \ ,\\
R^R_{xz} &= \frac{n}{b}J^2_n(b)\cos\psi \ ,\\
R^I_{xz} &= J_n(b)J'_n(b)\sin\psi \ ,\\
R^R_{yy} &= \frac{n^2}{b^2}J^2_n(b) + \cos^2\psi\left(J'^2_n(b) - \frac{n^2}{b^2}J^2_n(b)\right) \ ,\\
R^I_{yy} &= 0 \ ,\\
R^R_{yz} &= \frac{n}{b}J^2_n(b)\sin\psi \ ,\\
R^I_{yz} &= -J_n(b)J'_n(b)\cos\psi \ ,\\
R^R_{zz} &= J^2_n(b) \ ,\\
R^I_{zz} &= 0 \ ,
\end{aligned}$$

with $R^R_{ij} = R^R_{ji}$ and $R^I_{ij} = -R^I_{ji}$ [5].

These expressions are similar to those obtained for the homogeneous part of the dielectric tensor, with terms appended to the components ε_{iy} and with an ϵ-dependent term in the resonant denominator [5]. The striking feature about this tensor, which shows the inadequacy of the approximated orbits utilized as characteristics, is that it does not satisfy Onsager symmetry relations and therefore does not describe correctly the wave-particle interaction, according to the arguments previously expounded. If this tensor is utilized in the dispersion relation, the outcome is that the imaginary part of the wave vector is also dependent on nonresonant terms, which do not describe the energy exchange between a wave and particles.

Another feature of this approximation which deserves comment is that the resonant denonimator is modified only by the addition of the drift term, proportional to $\sin\psi$, as compared to the resonant denominator in the homogeneous case. Therefore, for perpendicular wave vector parallel to the magnetic field gradient there is no effect due to the inhomogeneity in contrast to the situation occurring when the nonlinear correction to the frequency is incorporated, as in the formulation developed in the present paper. This point can be verified by inspection of the resonant denominator of Eq. (12), in which it is seen that the term originated from the correction to the frequency is proportional to p_y and not only gives a contribution even for $\psi = 0^0$, but is also responsible for the φ dependence of the resonant denominator.

B. Unperturbed orbits with the nonlinear correction to the frequency, without taking into account the BGI corrections

In order to continue with the comparison between the symmetry properties of the effective dielectric tensor given by (23) and other approaches to the study of the dielectric properties of plasmas in inhomogeneous magnetic fields which appear in the literature, we now discuss another version for the dielectric tensor components, which we will call approach B [12,13,17,18]. The procedure employed for the derivation of the dielectric properties of the plasma in these papers has been based on the gyrokinetic theory [19,20] and is quite different from the one utilized in the present paper and developed in Sec. II. However, the same results can be obtained by the introduction of several simplifications in our derivation of the auxiliary tensor $\overleftrightarrow{\varepsilon}^0$. This we demonstrate by taking the derivation of the component ε^0_{zz} as an example.

We start from Eqs. (7) and (12) and then make several simplifying assumptions. We consider a Maxwellian distribution function, and therefore $\mathcal{L}(F_0) = 0$, and $\Phi_0(F_0) = \partial_\perp F_0$. We also neglect relativistic effects, making $\gamma = 1$, and consider the particular case $k_\parallel = 0$ and $\psi = 0$. Therefore, the component ε^0_{zz} can be written as

$$\varepsilon^0_{zz} = 1 + X\frac{\omega}{n_e}\sum_{n=-\infty}^{+\infty}\int d^3v\ v_\perp\left(\frac{v_\parallel}{v_\perp}\right)^2 \times\frac{e^{ik_\perp v_\perp \sin\varphi/\Omega}\,e^{-in\varphi}}{\omega - n\Omega(x) - n\epsilon v_\perp\sin\varphi}\,\partial_{v_\perp}F_0\,J_n(k_\perp v_\perp/\Omega) \ . \tag{29}$$

The assumed Maxwellian distribution will be denoted by

$$F_0 = \frac{n_e}{\pi^{3/2}v_T^3}e^{-v^2/v_T^2} \ , \tag{30}$$

where T is the electron temperature and $v_T = \sqrt{2T/m}$. Using this distribution function, the dielectric tensor given by Eq. (29) can be written as

$$\varepsilon^0_{zz} = 1 - \frac{X\omega}{\pi} \sum_{n=-\infty,\neq 0}^{+\infty} \frac{1}{n\epsilon v_T} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \times \frac{e^{ik_\perp \rho v_y}\, e^{-in\varphi}}{\frac{\omega - n\Omega(x)}{n\epsilon v_T} - v_y} e^{-(v_x^2+v_y^2)} J_n(k_\perp \rho v_\perp) - \frac{X}{\pi} \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y\, e^{ik_\perp \rho v_y}\, e^{-(v_x^2+v_y^2)} \times J_0(k_\perp \rho v_\perp)\ , \tag{31}$$

where the velocities have been normalized to the thermal velocity v_T and ρ is the electron Larmor radius $\rho \equiv v_T/\Omega$.

We now introduce another approximation: We consider that for wave frequencies near the electron cyclotron frequency only harmonics $n = 0, \pm 1$ are important. We also assume a small Larmor radius, so that $J_0 \simeq 1$, $J_1(x) \simeq x/2$, and $J_{-1} \simeq -x/2$. After some algebraic manipulation, Eq. (31) can be written as

$$\varepsilon^0_{zz} = 1 - X + i\frac{X\omega k_\perp \rho}{2\epsilon v_T}\left[-z_1 Z\left(z_1 - i\frac{k_\perp \rho}{2}\right) + z_{-1} Z\left(z_{-1} - i\frac{k_\perp \rho}{2}\right)\right] e^{-\frac{k_\perp^2 \rho^2}{4}}\ , \tag{32}$$

where $z_n \equiv \frac{\omega - n\Omega(x)}{n\epsilon v_T}$.

This expression for the zz component of the dielectric tensor, obtained from our ε^0_{zz} after several approximations, corresponds to Eq. (57) of [17]. It is utilized in that publication in order to discuss ordinary mode waves propagating perpendicularly across the fundamental electron cyclotron resonance [17]. The outcome is that the imaginary part of the wave vector, connected to the anti-Hermitian part of ε^0_{zz}, is obtained as a solution of the dispersion relation and has contributions which depend on the nonresonant electron response [17]. Since these are not connected with energy dissipation by the wave, they are neglected in the evaluation of the optical depth [17], although they can affect the local value of the absorption coefficient [14,17].

A formalism similar to the one utilized to discuss the O mode by Lashmore-Davies and Dendy [17] has been employed to study ion cyclotron damping, for propagation both perpendicular and oblique to the magnetic field [13,18]. For these waves, other components of the dielectric tensor have been evaluated, also lacking the adequated symmetry properties. This is not surprising since these components can be obtained from our ε^0_{ij} components after several simplifying assumptions, as we have seen in the case of the zz component. The lack of symmetry exhibited by the dielectric tensor derived with the utilization of approach B has as a consequence the nonresonant contribution to the imaginary part of the wave vector, which does not appear in the formulation developed in the present paper.

Another discussion of electron cyclotron resonance heating in an inhomogeneous medium utilizing orbits which include the nonlinear correction to the frequency has recently appeared in the literature [14] and generalizes for the relativistic case some results previously obtained in the nonrelativistic approximation [17]. The study has been developed using the wave equation formalism, in a rather straightforward way [14]. However, BGI corrections are not utilized and therefore the relationship between the current density and the electric field in $(\vec{k}, \omega)$ space is made by a conductivity tensor which does not feature Onsager symmetry and is not the Fourier transform of the dielectric tensor in configuration space. As a consequence, the dielectric tensor that can be obtained by means of this formalism is equivalent to our $\overleftrightarrow{\varepsilon}^0$ tensor. This point can be demonstrated quite easily by taking the ε^0_{zz} component from Eq. (19). By assuming a nonrelativistic Maxwellian distribution as given by Eq. (13) of Ref. [14], one arrives at the zz component of Eq. (15) of Ref. [14].

C. Final remarks about the comparison with other approaches from the literature

First of all, it is possible to conclude that the approximated orbits given by Eq. (26) are not adequate to be utilized in the derivation of the dielectric tensor. The important term to be kept is the nonlinear correction to the frequency. However, introduction of this correction is not enough to ensure the correctness of the dielectric tensor, as we have seen in the derivation of (32). It has to be coupled with the derivation of the effective dielectric tensor, following the procedure described by BGI, in order to get a tensor with the correct symmetry properties to describe wave-particle interactions in inhomogeneous magnetic fields. The importance of the nonlinear modification of the frequency has already been recognized in the literature, with some discussion about its consequences [11–14,17,18]. However, the connection with the symmetry properties of the dielectric tensor had not until now been adequately discussed, to the best of our knowledge, and a useful, complete, and general form of the dielectric tensor components satisfying the required symmetry conditions was not yet available.

V. ABSORPTION OF ORDINARY MODE WAVES FOR PERPENDICULAR PROPAGATION

In order to investigate quantitative effects due to the inhomogeneous magnetic field in the absorption of electromagnetic waves, we consider the case of perpendicularly propagating waves with frequency near the electron cyclotron frequency, in a plasma with a Maxwellian electron distribution function. For a distribution function that is even in $p_\parallel$, as in the case of a Maxwellian distribution, the dispersion relation factorizes into two branches, named the ordinary and extraordinary mode branches, respectively,

$$N_\perp^2 = \varepsilon_{zz}\ ,$$

$$N_\perp^2\left[\varepsilon_{xx}\cos^2\psi + \varepsilon_{yy}\sin^2\psi + (\varepsilon_{xy} + \varepsilon_{yx})\sin\psi\cos\psi\right] = (\varepsilon_{xx}\varepsilon_{yy} - \varepsilon_{xy}\varepsilon_{yx})\ , \tag{33}$$

where $N_\perp \equiv ck_\perp/\omega$ is the refraction index.

The Maxwellian distribution function for the electrons can be written as follows, as a function of the components of $\mathbf{u} = \mathbf{p}/(mc)$:

$$F_0(u_\perp^2, u_\parallel) = n_e \left(\frac{\mu}{2\pi}\right)^{3/2} e^{-\mu u^2/2} , \tag{34}$$

where n_e is the electron density and $\mu = m_e c^2/T_e$, where T_e is the electron temperature; considering a weakly relativistic regime $\gamma \simeq 1 + u^2/2$, $\theta = \pi/2$ and wave propagation along the direction of the inhomogeneity ($\psi = 0$), from Eq. (23) we arrive at the following expression for the effective dielectric tensor components:

$$\varepsilon_{ij} = \delta_{ij} + i2\pi X\omega\mu \left(\frac{\mu}{2\pi}\right)^{3/2} \sum_{n=-\infty}^{\infty} (-1)^{n+1-\delta_{i3}} \int_0^\infty d\tau\, e^{i[\omega - n\Omega(x)]\tau}\, e^{-in\theta'_n}$$
$$\times \int_{-\infty}^{\infty} du_\parallel\, e^{-\frac{\mu}{2}u_\parallel^2(1-i\frac{\omega}{\mu}\tau)} \int_0^\infty du_\perp u_\perp^3\, e^{-\frac{\mu}{2}u_\perp^2(1-i\frac{\omega}{\mu}\tau)}\, e^{-i\frac{\epsilon b}{2m}p_\perp \sin\psi\tau} \pi_i^{-n}\pi_j^n . \tag{35}$$

The notation can be simplified by the definition of the following quantities:

$$t \equiv \frac{\omega}{\mu}\tau ,$$
$$\Delta_n \equiv \mu[1 - n\Omega(x)/\omega] , \tag{36}$$
$$\alpha \equiv 1 - \frac{c}{\omega}\epsilon\mu^{1/2}\beta\sin\psi , \tag{37}$$

where $\beta \equiv N_\perp/(Y\mu^{1/2})$ and $Y \equiv \Omega/\omega$. Equation (35) can then be written as

$$\varepsilon_{ij} = \delta_{ij} + i2\pi X\mu^2 \left(\frac{\mu}{2\pi}\right)^{3/2} \sum_{n=-\infty}^{\infty} (-1)^{n+1-\delta_{i3}}$$
$$\times \int_0^\infty dt\, e^{i\Delta_n t}\, e^{-in\theta'_n}$$
$$\times \int_{-\infty}^{\infty} du_\parallel\, e^{-\frac{\mu}{2}u_\parallel^2(1-it)}$$
$$\times \int_0^\infty du_\perp\, u_\perp^3\, e^{-\frac{\mu}{2}u_\perp^2(1-i\alpha t)}\pi_i^{-n}\pi_j^n . \tag{38}$$

By performing the momentum integrals, we arrive at the following expression for the components of the dielectric tensor:

$$\varepsilon_{ij} = \delta_{ij} + \sum_{n=1}^{\infty}\sum_{s=\pm 1} \varepsilon_{ij}^{ns} + \varepsilon_{ij}^{00} , \tag{39}$$

where

$$\varepsilon_{ij}^{ns} = -iX\mu^2 \int_0^\infty dt\, \frac{e^{i\Delta_{ns}t}\, e^{-in\theta'_n}}{\zeta_{ns}\zeta_{-ns}(1-it)^{1/2}(1-i\alpha t)}$$
$$\times\Bigg\{ M_{ns}\left[\frac{a_{ij}^0}{\zeta_{ns}\zeta_{-ns}} + \frac{a_{ij}^1}{\zeta_{ns}\zeta_{-ns}(1-i\alpha t)} + \frac{\zeta_{ns}\zeta_{-ns}a_{ij}^2}{(1-i\alpha t)^2}\right]$$
$$+ N_{ns}\left[\frac{b_{ij}^1}{(1-i\alpha t)} + \frac{b_{ij}^2}{(1-i\alpha t)^2}\right]\Bigg\} ,\quad i,j = 1,2$$
$$\varepsilon_{33}^{ns} = iX\mu \int_0^\infty dt\, \frac{e^{i\Delta_{ns}t}\, e^{-in\theta'_n}}{(1-it)^{3/2}(1-i\alpha t)} M_{ns} ,$$
$$\varepsilon_{13}^{ns} = \varepsilon_{31}^{ns} = \varepsilon_{23}^{ns} = \varepsilon_{32}^{ns} = 0 ,$$

and $\Delta_{ns} \equiv \mu[1 - ns\Omega(x)/\omega]$. The quantities M_{ns} and N_{ns} result from the momentum integrals and are dependent upon the modified Bessel function of the first kind, the I_n function [21],

$$M_{ns} = \exp\left(-\frac{\beta^2(\zeta_{ns} - \zeta_{-ns})^2}{2(1-i\alpha t)}\right)$$
$$\times\exp\left(-\frac{\beta^2\zeta_{ns}\zeta_{-ns}}{1-i\alpha t}\right) I_n\left(\frac{\beta^2\zeta_{ns}\zeta_{-ns}}{1-i\alpha t}\right) ,$$
$$N_{ns} = \exp\left(-\frac{\beta^2(\zeta_{ns} - \zeta_{-ns})^2}{2(1-i\alpha t)}\right)$$
$$\times\exp\left(-\frac{\beta^2\zeta_{ns}\zeta_{-ns}}{1-i\alpha t}\right) I'_n\left(\frac{\beta^2\zeta_{ns}\zeta_{-ns}}{1-i\alpha t}\right) .$$

The a_{ij}^k and b_{ij}^k coefficients, which appear in Eq. (39), are given by

$$a_{11}^0 = -\frac{n^2}{\mu\beta^2}\left[1 - \left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^2\right] ,$$
$$a_{12}^0 = \frac{n^2}{\mu\beta^2} 2\sin\psi\left(\frac{\hat\epsilon_{ns}t}{\beta}\right) ,$$
$$a_{21}^0 = -a_{12}^0,\quad a_{22}^0 = a_{11}^0 ,$$
$$a_{11}^1 = i\frac{2ns}{\mu}\sin\psi\left(\frac{\hat\epsilon_{ns}t}{\beta}\right)\left[1 - 2\cos^2\psi + \left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^2\right] ,$$
$$a_{12}^1 = i\frac{ns}{\mu}\left[1 + 2\cos\psi(\cos^2\psi - \sin^2\psi)\left(\frac{\hat\epsilon_{ns}t}{\beta}\right) - 2\cos\psi\left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^3 - \left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^4\right] ,$$
$$a_{21}^1 = -i\frac{ns}{\mu}\left[1 - 2\cos\psi(\cos^2\psi - \sin^2\psi)\left(\frac{\hat\epsilon_{ns}t}{\beta}\right) + 2\cos\psi\left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^3 - \left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^4\right] ,$$
$$a_{22}^1 = i\frac{2ns}{\mu}\sin\psi\left(\frac{\hat\epsilon_{ns}t}{\beta}\right)\left[1 + 2\cos^2\psi + \left(\frac{\hat\epsilon_{ns}t}{\beta}\right)^2\right] ,$$
$$a_{11}^2 = -\frac{2\beta^2}{\mu}\sin^2\psi ,$$

$$a_{12}^2 = \frac{2\beta^2}{\mu}\sin\psi\left(\cos\psi + \frac{\hat{\epsilon}_{ns}t}{\beta}\right) ,$$
$$a_{21}^2 = \frac{2\beta^2}{\mu}\sin\psi\left(\cos\psi - \frac{\hat{\epsilon}_{ns}t}{\beta}\right) ,$$
$$a_{22}^2 = -\frac{2\beta^2}{\mu}\left[\cos^2\psi - \left(\frac{\hat{\epsilon}_{ns}t}{\beta}\right)^2\right] ,$$
$$b_{11}^1 = -i\frac{2ns}{\mu}\sin\psi\left(\frac{\hat{\epsilon}_{ns}t}{\beta}\right) ,$$
$$b_{12}^1 = -i\frac{ns}{\mu}\left[1 - \left(\frac{\hat{\epsilon}_{ns}t}{\beta}\right)^2\right] ,$$
$$b_{21}^1 = -b_{12}^1,\ b_{22}^1 = b_{11}^1 ,$$
$$b_{ij}^2 = -\left[1 + \left(\frac{\hat{\epsilon}_{ns}t}{\beta}\right)^2\right] a_{ij}^2,$$

where we have defined the quantity $\hat{\epsilon}_{ns} = ns\epsilon c\mu^{1/2}/(2\omega)$. In the derivation of Eq. (39) we have introduced the symbol s for the sign of n.

In the homogeneous case ($\epsilon = 0$), the ε_{ij}^{ns} quantities, which appear in Eq. (39), are equivalent to Eq. (62) of Ref. [22] and may be expressed in terms of the relativistic plasma dispersion functions $\mathcal{F}_q$. In the inhomogeneous case, Eq. (39) can also be expanded in terms of these well-known plasma dispersion functions, but we think that the explicit expansion does not contribute to simplifying the appearance or the use of our expressions. However, by the use of some restrictive approximations, we can write the effective dielectric tensor in terms of the very familiar plasma dispersion function Z.

Let us consider as an example the case of ε_{zz}. Inserting the distribution (34) into Eq. (23), using a nonrelativistic approximation ($\gamma = 1$), and taking into account only the $n = 0, \pm 1$ terms in the expansion, we arrive at the following zz component, after performing the u-space integrals:

$$\varepsilon_{33} = 1 - Xe^{\beta^2} I_0(\beta^2) + i\mu X e^{-\beta^2} \times \sum_{s=\pm 1}\int_0^\infty dt\, e^{i\mu[1-sY(x)]t} e^{-\hat{\epsilon}_{1s}^2 t^2} I_n(\beta^2 - \hat{\epsilon}_{1s}^2 t^2) . \tag{40}$$

It is possible to separate the argument of the I_n function by using an addition theorem for Bessel functions [21]. Using a small Larmor radius approximation for the first term and the lowest-order term in the expansion for the second, we arrive at the following result:

$$\varepsilon_{33} = 1 - X + \frac{1}{4}\mu X\beta^2 \sum_{s=\pm 1}\frac{1}{|\hat{\epsilon}_{1s}|} Z\left(\frac{\Delta_s}{2\hat{\epsilon}_{1s}}\right) , \tag{41}$$

where

$$Z(\chi) = 2\,i\,e^{-\chi^2}\int_{-\infty}^{i\chi} dy\, e^{-y^2}$$

is the well-known Fried-Conte plasma dispersion function.

This result shows that the anti-Hermitian part of ε_{zz} is connected only to the imaginary part of $Z(\chi)$ and therefore only to the resonating particles. The remaining terms of the expansion of the Bessel functions also display the same property. It is interesting to compare this result with Eq. (32), obtained by the use of approach B, whose anti-Hermitian part has contributions due to nonresonant particles [17].

For the numerical investigation, we consider ordinary mode waves propagating perpendicularly to $\mathbf{B}_0$ in the equatorial plane of a tokamak described by a plasma slab, with profiles given by

$$n_e(x) = n_e(0)\left(1 - \frac{x^2}{a^2}\right)^p,$$
$$T_e(x) = T_e(0)\left(1 - \frac{x^2}{a^2}\right)^q, \tag{42}$$
$$B_0(x) = B_0(0)\left(1 - \frac{x}{R}\right),$$

where a is the radius of the plasma column and R is the torus radius. Therefore, the waves are propagating along the direction of the inhomogeneity ($\sin\psi = 0$). According to our previous definitions, the quantity ϵ is therefore given by $\epsilon \equiv -R^{-1}$.

Initially, we assume JET-like parameters (JET: Joint European Torus), with central electron density $n_e(0) = 1 \times 10^{14}$ cm^{-3}, central electron temperature $T_e(0) = 5.0$ keV, magnetic field at the center $B_0(0) = 3.4$ T, $a = 120$ cm, and consider R ranging from $R \simeq a$ to $R \to \infty$, that is, from a small aspect ratio to a very large aspect ratio, in order to study the effect of different degrees of inhomogeneity on the ordinary mode absorption (the aspect ratio is defined as $\rho = R/a$). We also choose simple profiles, with $p = 1$ and $q = 2$. We then assume a real wave frequency equal to the electron cyclotron frequency at position $x = 0$ cm, launched from the external side of the tokamak in the equatorial plane, with $k_\parallel = 0$. We solve numerically the ordinary mode branch of the dispersion relation (33), obtaining the complex quantity $N_\perp$. Figure 1 shows the real and imaginary parts of $N_\perp$ for the ordinary mode, respectively, denoted by $\mathrm{Re}(N_O)$ and $\mathrm{Im}(N_O)$, as functions of position, for three values of R, corresponding to aspect ratio values $\rho = 2.58$, $\rho = 2.00$, and $\rho = 1.50$. For each case the locally homogeneous result, obtained with the use of $\epsilon = 0$, is shown as a full line and the inhomogeneous result is shown as a broken line. As the aspect ratio is decreased the absorbing region moves toward the central position of the slab, as expected. The comparison between the homogeneous case and the inhomogeneous cases shows that the inhomogeneity effect is practically irrelevant for these parameters, even in the most inhomogeneous situation of very small aspect ratio. The reasons for this feature can be qualitatively understood by the following considerations.

For the geometry considered, $\sin\psi = 0$, and therefore the quantity θ_n' vanishes, α becomes 1, and M_{ns} can be written as

$$M_{ns} = \exp\left(-\frac{\beta^2 + n^2c^2Y^2\mu\epsilon^2t^2/(4\Omega^2)}{(1-it)}\right) \times I_n\left(\frac{\beta^2 - n^2c^2Y^2\mu\epsilon^2t^2/(4\Omega^2)}{1-it}\right) . \quad (43)$$

As a consequence, the zz component of the dielectric tensor, given by (39), depends on the integral

$$\int_0^\infty dt\, \frac{e^{i\mu(1-nsY-nsY\epsilon x)t}}{(1-it)^{5/2}} \times \exp\left(-\frac{\beta^2 + n^2c^2Y^2\mu\epsilon^2t^2/(4\Omega^2)}{(1-it)}\right) \times I_n\left(\frac{\beta^2 - n^2c^2Y^2\mu\epsilon^2t^2/(4\Omega^2)}{1-it}\right) . \quad (44)$$

It is seen that inhomogeneity-related factors appear in the arguments of the exponential function and the modified Bessel function. These arguments can be written as

$$\beta^2\left[1 \pm \frac{n^2}{4}\frac{c^2Y^2}{N_\perp^2}\frac{\mu^2\epsilon^2}{\Omega^2}\right] \simeq \beta^2\left[1 \pm \frac{1.9\times10^3\,t^2}{R^2B^2T^2}\right] . \quad (45)$$

The right-hand side of (45) has been written by considering that, for the spatial region where cyclotron absorption is meaningful, $Y \simeq 1$ and the real part of $N_\perp$ is also near unity. Equation (45) is written for R measured in centimeters, T in keV, and B in tesla. For the parameters utilized for Fig. 1, namely, $B_0 = 3.4$ T, $T_e(0) = T = 5$ keV, and $R = 120$ cm, (45) reduces to

$$\beta^2\left[1 \pm 4.57\times10^{-4}\,t^2\right] . \quad (46)$$

The integrand in Eq. (44) is quickly convergent, being significant only for values of t such that $t \leq 2$. Therefore, Eq. (46) shows that in the region of t values for which the integrand of Eq. (44) is meaningful, the arguments of the exponential and Bessel functions are very close to the values for the homogeneous case. However, for smaller values of the temperature or the magnetic field intensity, the effect of the inhomogeneity may become appreciable. Consider, for instance, the case of $B_0 = 1.5$ T, $T = 0.5$ keV, and $R = 100$ cm, for which (45) is given by

$$\beta^2\left[1 \pm 0.34\,t^2\right] , \quad (47)$$

where it is seen that for values of t which are relevant for the integral, the inhomogeneity contribution for the argument has become meaningful.

As an illustration, we assume parameters in the range typical of small tokamaks, with $B_0(0) = 1.5$ T, $T_e(0) = 0.5$ keV, $n_e(0) = 1\times10^{13}$ cm^{-3}, $a = 20$ cm, $p = 1$, $q = 2$, and consider R ranging from $R \simeq a$ to $R \to \infty$. In the small aspect ratio limit, these parameters are similar to those expected for START (Small Tight Aspect Ratio Tokamak) and other small aspect ratio devices already in operation or planned for the near future. As in the case of Fig. 1, we choose the wave frequency to be equal to the electron cyclotron frequency at position $x = 0$ cm, launched from the external side of the tokamak with $k_\parallel = 0$, and solve numerically the ordinary mode branch of the dispersion relation (33). Figure 2 shows the outcome of these calculations, in the form of the real and imaginary parts of N_O as functions of position, for

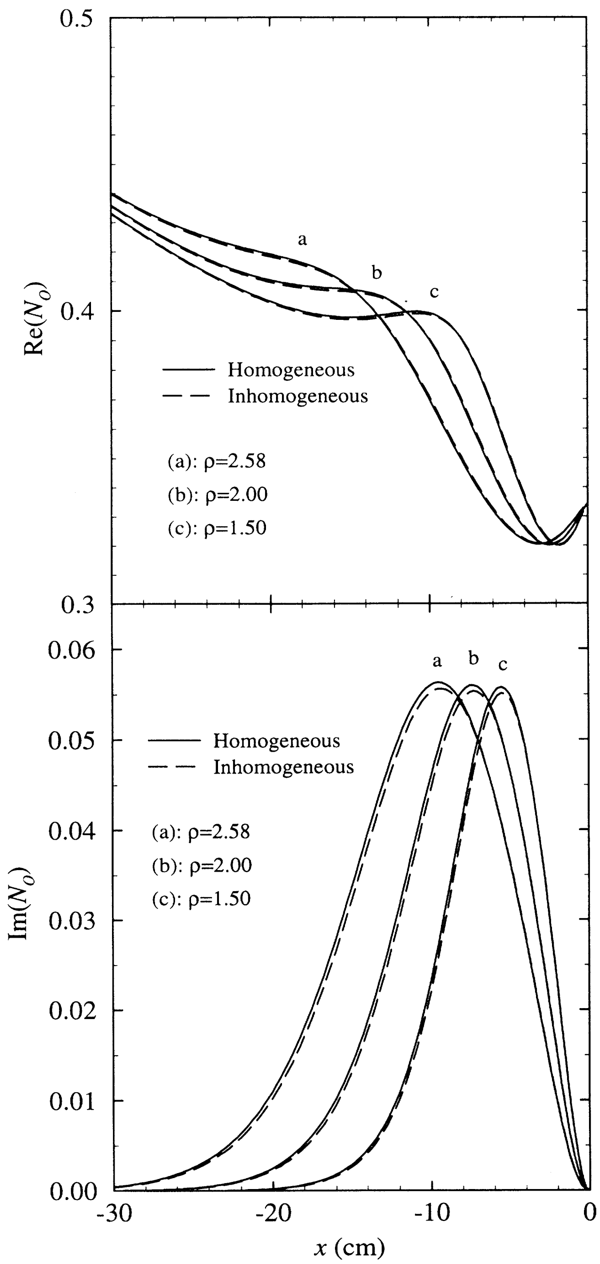

FIG. 1. Real and imaginary parts of the refraction index for the ordinary mode, Re(N_O) and Im(N_O), vs position in the plasma slab, for three values of the aspect ratio and for $\theta = \pi/2$ and $\psi = 0$; $B_0 = 3.4$ T, $T = 5.0$ keV, $n_e = 1\times10^{14}$ cm^{-3}, $a = 120$ cm, and the wave angular frequency $\omega = \Omega(0)$. "Homogeneous" results are obtained with the use of the dielectric tensor for locally homogeneous plasmas, and "inhomogeneous" results are obtained with the use of the effective dielectric tensor.

three values of R, corresponding to aspect ratio values of $\rho = 4.5$, 4.0, and 3.0. The inhomogeneous result, obtained with the use of the effective dielectric tensor, is shown by the use of bold lines in Fig. 2, labeled as (I), while the corresponding result obtained with the locally homogeneous tensor is shown by weaker lines labeled as (H). It is seen that for these parameters of temperature and magnetic field the local value of the absorption coefficient for the ordinary mode is substantially modified as compared to the homogeneous result. An interesting feature emerges, which is the appearance of a region of negative absorption coefficient near the electron cyclotron frequency, for sufficiently large inhomogeneity, which is equivalent in the present case of a tokamak profile to a sufficiently small aspect ratio.

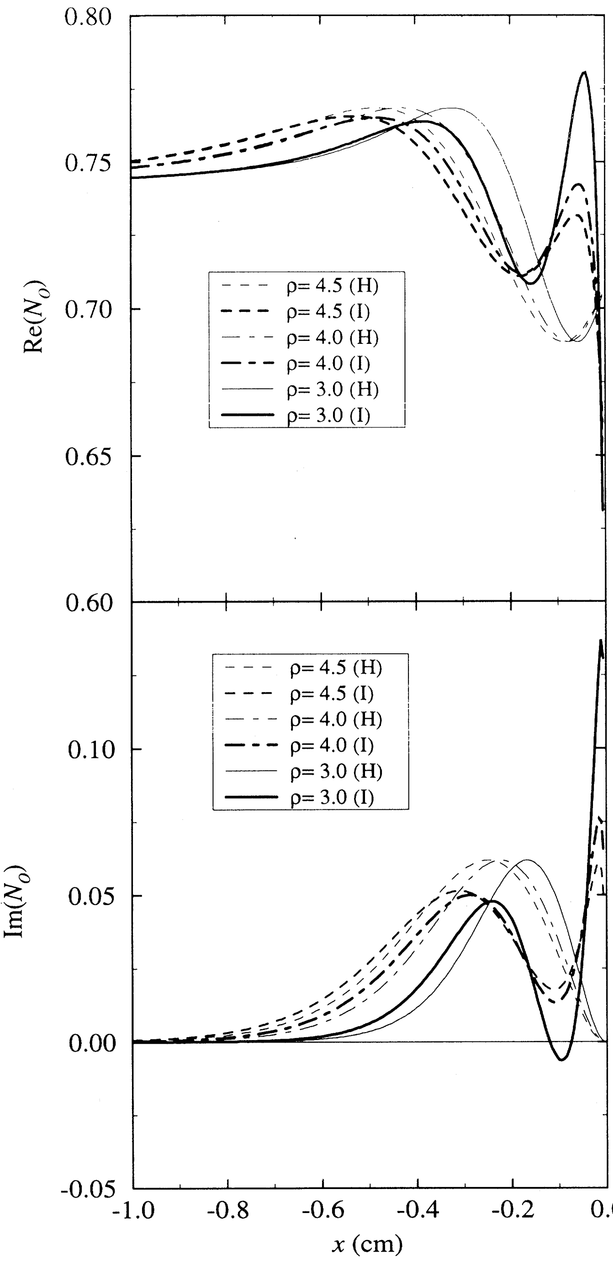

FIG. 2. Real and imaginary parts of the refraction index for the ordinary mode, $\mathrm{Re}(N_O)$ and $\mathrm{Im}(N_O)$, vs position in the plasma slab, for three values of the aspect ratio and for $\theta = \pi/2$ and $\psi = 0$; $B_0 = 1.5$ T, $T = 0.5$ keV, $n_e = 1 \times 10^{13}$ cm^{-3}, $a = 20$ cm, and the wave angular frequency $\omega = \Omega(0)$. The curves identified by the label (H) are obtained with the use of the dielectric tensor for locally homogeneous plasmas, while those identified by the label (I) are obtained with the use of the effective dielectric tensor.

The dependence of the inhomogeneity effect on temperature, depicted in Eq. (45), may appear to be surprising. The inhomogeneity effect actually increases when the temperature is decreased, although one could expect the contrary, because the Larmor radius decreases with the temperature, and the inhomogeneity is "felt" by the particles as they go along their cyclotron orbits. However, a simple qualitative argument helps to explain this apparently puzzling feature.

The exchange of energy between waves and particles is dependent upon the derivative of the distribution function, namely, upon $\partial_{u_\perp} F_0$. The maximum value of this derivative occurs for $u_\perp = \sqrt{1/\mu}$. Using Eq. (34), it is easily found that the maximum value of the relevant derivative is proportional to T^{-2}. Therefore, one can define a typical scale length of population variation in u space, by writing $\partial_{u_\perp} F_0 \mid_{\max} \simeq \Delta F_0 / L_u$, and it is seen that the temperature dependence is such that $L_u \sim T^2$.

On the other hand, the change in resonant momentum experienced by a displacement equivalent to the Larmor radius can be given by $\Delta u_{\mathrm{res}} \simeq \mid \Omega \mid r_L$ and its dependence in temperature is such that $\Delta u_{\mathrm{res}} \sim \sqrt{T}$. Therefore, along the orbit, the ratio between the change in resonant momentum and the change in particle population can be measured by $\Delta u_{\mathrm{res}} / L_u \sim T^{-3/2}$. The conclusion is that, when the temperature is decreased, the particle excursion is indeed decreased and similarly for the change in resonant momentum; but the change in particle population in momentum space experienced along the resonance curve is increased to a larger extent and this is the really relevant quantity for the wave-particle energy exchange.

We now investigate the effect of the inhomogeneity over a nonlocal quantity as the integrated absorption, by assuming that ordinary mode waves with frequency equal to the electron cyclotron frequency at position $x = 0$ cm are launched from the external side of the tokamak with $k_\parallel = 0$. The integrated absorption is defined as

$$\eta(x) = 1 - \exp\left(-2\frac{\omega}{c}\int_{x_0}^{x} ds\, \mathrm{Im}(N_\perp)\right) , \qquad (48)$$

where x_0 is the wave starting position and s is the integration variable along the wave path. The integrated absorption is shown in Fig. 3 for the same parameters utilized for the case depicted in Fig. 2, for three values of R. We see that the integrated absorption does not appear to be very sensitive to the influence of the inhomogeneity on the absorption coefficient. Even for a sufficiently small aspect ratio, the region of negative absorption which has been demonstrated in Fig. 2 to exist near the electron cyclotron frequency is compensated by increased absorption in another range of frequencies, with

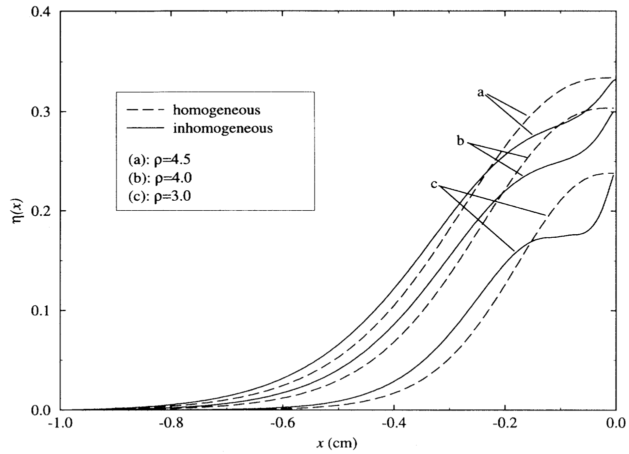

FIG. 3. Integrated absorption $\eta(x)$ for the ordinary mode vs position in the plasma slab, for the same values of the aspect ratio as in Fig. 2; other parameters are the same as in Fig. 2.

the result that the integrated absorption does not depart very much from the prediction of the homogeneous case.

These results, particularly those of Fig. 2, have shown that the magnetic field inhomogeneity is effective near the electron cyclotron frequency, where the absorbing particles are in the body of the distribution function. For this range of frequencies, the inhomogeneity effects may compete with relativistic effects, while further away from the cyclotron frequency relativistic effects are clearly dominant and the effect of the inhomogeneity is negligible. It may be remarked that in the case of density and temperature inhomogeneities the inhomogeneity effects were more noticeable for down-shifted frequencies, mainly absorbed by electrons in the tail of the distribution function [7].

We now compare the results predicted by the use of the effective dielectric tensor given by (39) with results obtained from other approaches to the problem of wave absorption in inhomogeneous plasmas. In order to do that, we solve the dispersion relation for the ordinary mode, as given by (33), with the use of different versions of the dielectric tensor. The results are shown in Fig. 4, where we plot the imaginary part of N_O as a function of position, for three different cases. Figure 4(a) has been obtained with the use of the same parameters utilized in Fig. 1 and $R = 180$ cm. The curve labeled (1) is the same as the broken line curve c in Fig. 1 and depicts $\mathrm{Im}(N_O)$ obtained with the use of the effective dielectric tensor, as given by (39). The curve labeled (2) shows the result obtained with the use of the tensor whose components are given by Eq. (19), the relativistic dielectric tensor prior to the introduction of the BGI corrections (as we have stressed, this tensor does not display the required symmetry properties). It is seen that near the electron cyclotron frequency a region of negative values of the absorption coefficient appears, similarly to what has been obtained for different parameters in a recent publication [14]. This negative value of the imaginary part of the refraction index is due to nonresonant terms in the dielectric tensor and does not mean true wave amplification [14]. Curve (3) shows the result obtained with the use of the approximated nonrelativistic tensor derived with the use of approach B, whose components are given by Eq. (32). Finally, curve (4) shows the result obtained with the use of the tensor whose components are given by Eq. (28), following approach A. As we have already mentioned, in the case of this version for the dielectric tensor, derived without the nonlinear correction to the frequency in the unperturbed orbits, the inhomogeneity effect disappears for $\psi = 0°$ and curve (4) in fact coincides with the homogeneous result.

From these results, as well as from those of Fig. 1, it is observed that the use of the correct effective dielectric tensor, with relativistic effects included, predicts very small effects of the inhomogeneity in the magnetic field for these parameters. The same is true when it utilized the tensor without the BGI corrections, the auxiliary tensor $\overleftrightarrow{\varepsilon}^0$. The description in this case is not correct, as we have seen, but it has in common with the correct description the fact that the inhomogeneity leads to small modifications as compared to the homogeneous result. This is in contrast to the results obtained with the use of the nonrelativistic approximated tensor of approach B, with components given by (32). The homogeneous limit cannot even be treated in the nonrelativistic approximation for the present case of perpendicular propagation and the inhomogeneity effect modifying the resonant denominator results in high values of the absorption coefficient, symmetric around cyclotron frequency. Therefore, it is illustrated that the effective dielectric tensor not only has adequated formal features, but when utilized in the dispersion relation predicts results quantitatively different from those obtained with other approaches utilized in the literature.

In Fig. 4(b), we consider the case of the same parameters utilized for Fig. 2, with $R = 90$ cm, namely, a case where the aspect ratio cannot be considered small and the inhomogeneity effect still does not lead to the appearance of the region of negative absorption coefficient. The curve labeled (1) has been obtained with the use of the effective dielectric tensor and is the same as the curve obtained with $\rho = 4.5$ in Fig. 2. The curve labeled (2) shows the result obtained with the use of the auxiliary tensor $\overleftrightarrow{\varepsilon}^0$ [Eq. (19)], curve (3) shows the result obtained with the use of the approximated nonrelativistic tensor derived according to approach B [Eq. (32)], and curve (4) shows the result obtained with the use of the tensor of approach A [Eq. (28)], being coincident with the homogeneous result. The modification of the absorption coefficient appearing with the use of the $\overleftrightarrow{\varepsilon}^0$ tensor in this case becomes very conspicuous and the region of negative absorption coefficient extending beyond the cyclotron frequency is clearly seen. As we have discussed, it is due to nonresonant terms and does not mean true wave amplification [14].

For the case depicted in Fig. 4(b), the quantitative differences between the results obtained with different approaches become much more meaningful than in the case of Fig. 4(a). There are regions where the inhomogeneous result given by curve (1) differs significantly from the homogeneous result [coincident with curve (4)]. An appreciable difference can also be noticed between curve (1) and curve (2), for which the energy exchange between a wave and particles is not correctly described, and an even more appreciable difference between curve (1) and curve (3), obtained with the use of Eq. (32) for the components of the dielectric tensor. It may be repeated here, for the sake of completeness, that when the components of the dielectric tensor are given by Eq. (28), there is no effect of the inhomogeneity predicted for the direction of propagation considered here.

In Fig. 4(c) we consider the case of the same parameters utilized for Fig. 2, with $R = 60$ cm. For this case we observe the appearance of the inhomogeneity driven negative absorption coefficient. The results are displayed in the same way as in Figs. 4(a) and 4(b). The curve labeled (1) has been obtained with the use of the effective dielectric tensor and is the same as the curve which shows the result obtained for $\rho = 3.0$ in Fig. 2. The curve labeled (2) shows the result obtained with the use of the tensor whose components are given by Eq. (19), curve (3) shows the result obtained with the use of the approximated nonrelativistic tensor whose components are given by Eq. (32), and curve (4) shows the result obtained with the use of the tensor whose components are given by Eq. (28), being coincident with the homogeneous result. Comments similar as those which apply to Fig. 4(b) are also applicable to Fig. 4(c), regarding the comparison between different approaches.

Therefore, these examples have shown that the utilization of three versions of the dielectric tensor whose components do not respect the correct symmetry, namely, those given by Eqs. (19), (32), and (28), lead to quite different results for the absorption coefficient. We have seen that the correct inclusion of inhomogeneity effects modifies the usual results for the absorption profile of the ordinary mode near the electron cyclotron frequency, where the inhomogeneity correction may compete with relativistic effects. The absorption grows near the electron cyclotron frequency. It continues to grow as the inhomogeneity is increased, acquiring some similarity to the absorption profile displayed by curve (3) in Figs. 4(a)–4(c), obtained with a nonrelativistic formulation. However, the absorption peak originating from the correct formulation does not appear to be symmetric around the electron cyclotron frequency, as obtained with approach B, because the relativistic effect precludes significant res-

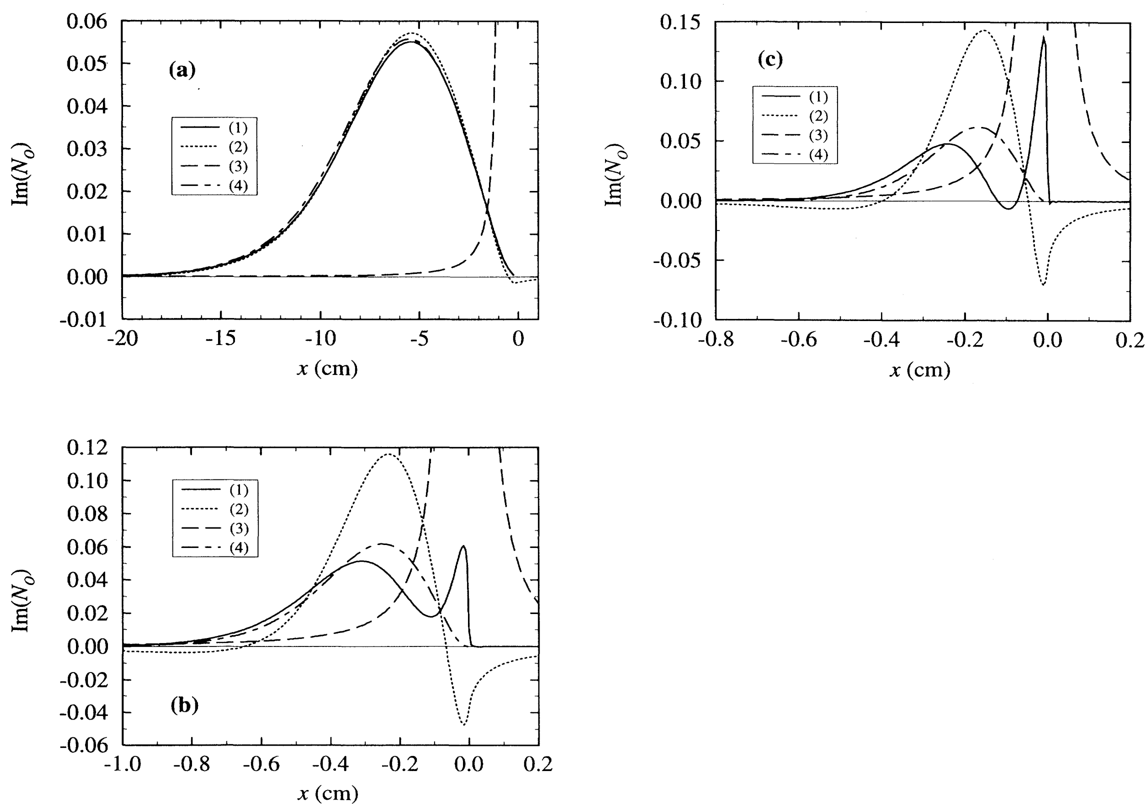

FIG. 4. Imaginary part of the refraction index for the ordinary mode vs position in the plasma slab, for $\omega = \Omega(0)$ and for $\theta = \pi/2$ and $\psi = 0$. (a) $B_0 = 3.4$ T, $T = 5.0$ keV, $n_e = 1 \times 10^{14}$ cm^{-3}, $a = 120$ cm, and $R = 180$ cm; (b) $B_0 = 1.5$ T, $T = 0.5$ keV, $n_e = 1 \times 10^{13}$ cm^{-3}, $a = 20$ cm, and $R = 90$ cm; (c) $B_0 = 1.5$ T, $T = 0.5$ keV, $n_e = 1 \times 10^{13}$ cm^{-3}, $a = 20$ cm, and $R = 60$ cm. In each panel the curves are labeled according to the following convention: (1) using the effective dielectric tensor given by the present formulation, (2) using the auxiliary tensor $\overleftrightarrow{\varepsilon}^0$, (3) using approach B, and (4) using approach A (which, in the present case, coincides with the homogeneous result).

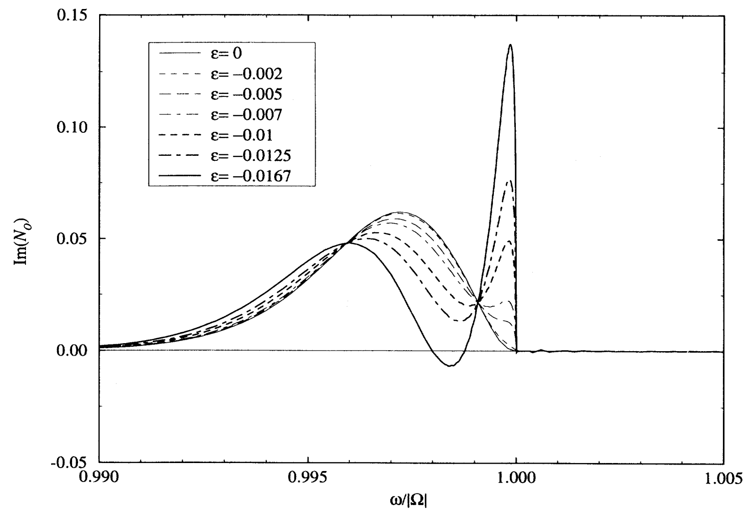

FIG. 5. Imaginary part of the refraction index for the ordinary mode, $\mathrm{Im}(N_O)$, vs $\omega/|\Omega|$, for several values of ϵ, evaluated with the use of the effective dielectric tensor in the dispersion relation. $B_0 = 1.5$ T, $T = 0.5$ keV, $n_e = 1 \times 10^{13}$ cm^{-3}, $\theta = \pi/2$, and $\psi = 0$.

onance above the cyclotron frequency, unless the inhomogeneity parameter ϵ is very large.

The features discussed above can also be illustrated outside the context of tokamak profiles, in order to isolate the effect of the inhomogeneity. In order to do that, we have solved the dispersion relation for the ordinary mode, with the use of the effective dielectric tensor, for the same plasma parameters as those describing the center position of the tokamak in the case of Fig. 2, namely, $B_0 = 1.5$ T, $T = 0.5$ keV, and $n_e = 1 \times 10^{13}$ cm^{-3}, and for $\theta = \pi/2$ and $\psi = 0$. The imaginary part of the refraction index resulting from this calculation is shown in Fig. 5 as a function of angular frequency, for several values of ϵ, starting from the homogeneous case ($\epsilon = 0$).

It has been seen in our examples that for sufficiently large inhomogeneity, a region of negative absorption coefficient may appear in a range of frequencies. This is a result that deserves further investigation. The imaginary part of the wave vector must be connected to actual wave-particle energy exchange since it was obtained from a dispersion relation with the use of the effective dielectric tensor; one can nonetheless verify whether the negative absorption indeed corresponds to wave amplification or whether it is related to the reversed sign in the group velocity. In order to do that, we have assumed real $k_\perp$ and complex ω in the ordinary mode branch of Eq. (33) and used the familiar approximated expression for the growth rates, valid for $|\omega_i| \ll \omega_r$,

$$\omega_i \simeq -D_i(\omega_r)/\partial_\omega D_r(\omega_r)\,, \tag{49}$$

where D_r and D_i are, respectively, the real and imaginary parts of the dispersion relation. The curve labeled ω_i (1) in Fig. 6 shows the imaginary part of ω vs $\omega_r/|\Omega|$, obtained according to these procedures, for the same parameters utilized in Fig. 5 and the same value of ϵ which resulted in negative absorption coefficient in Fig. 5, namely, $\epsilon = -0.0167$. It is seen that there is a region of positive values of ω_i, in the same frequency interval in which Fig. 5 exhibits a negative value of the absorption coefficient. We also display in Fig. 6, by curve labeled

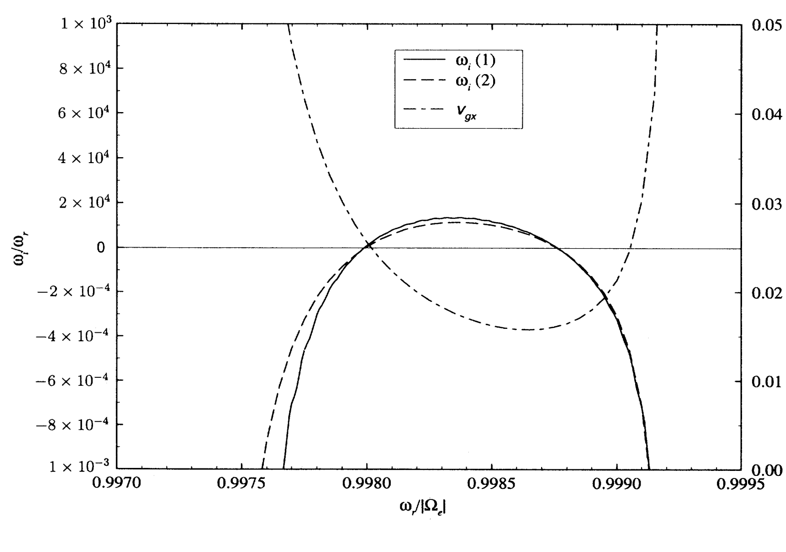

FIG. 6. ω_i/ω_r vs $\omega_r/|\Omega|$, for the ordinary mode, for the same parameters utilized in Fig. 5, and $\epsilon = -0.0167$. ω_i (1), result obtained using $\omega_i \simeq -D_i(\omega_r)/\partial_\omega D_r(\omega_r)$; ω_i (2), result obtained with the approximated expression $\omega_i \simeq -k_i v_{gx}$ (k_i is numerically obtained from the dispersion relation).

ω_i (2), the result of an alternative procedure, which assumes that the growth rates are approximately given by $\omega_i \simeq -k_i v_{gx}$, where v_{gx} is the x component of the group velocity. It is seen that the result obtained with this second method predicts a frequency interval of unstable waves which coincides with the interval obtained with the use of Eq. (49), with close agreement of magnitude. Figure 6 also displays the numerically evaluated group velocity for the same frequency range. It is seen that the interval of unstable waves occurs in a region of well-behaved group velocity ($v_{gx} < c$) and does not coincide with regions of anomalous dispersion, where the group velocity changes sign.

Therefore, it is verified that the negative absorption coefficient obtained for sufficiently large inhomogeneity is not due to the reversed sign of the group velocity. In fact, because of the correct symmetry properties of the effective dielectric tensor, its anti-Hermitian part is only due to resonant terms and this ensures that the imaginary part of the refraction index really describes energy dissipation or energy growth. This is in contrast to the results obtained with other formulations, featuring negative values of the absorption coefficient which are due to nonresonant terms in the dielectric tensor.

Finally, it is interesting to note that in a recent publication it is stated that the correction in the cyclotron frequency due to the inhomogeneity does not compete efficiently with the relativistic correction and that the effect due to the inhomogeneity in the absorption of perpendicularly propagating waves is negligible in all cases of practical interest [23]. We have seen that the use of the effective dielectric tensor confirms this statement for parameters typical of present large tokamaks. However, by exploring other parametric ranges, it has been shown that significant inhomogeneity effects on the local absorption coefficient of ordinary mode waves may be present under laboratory conditions, although with negligible effect on the optical depth.

VI. CONCLUSIONS

In the present paper we have studied the dielectric properties of plasmas in weakly inhomogeneous magnetic fields by considering the case of inhomogeneity in the direction perpendicular to the ambient magnetic field. We have derived the unperturbed orbits for such a geometry and the components of the dielectric tensor which must be utilized when solving the dispersion relation. This dielectric tensor is an effective tensor, derived according to an iterative procedure proposed by Beskin, Gurevich, and Istomin [6]. The BGI procedure has been devised in order to ensure energy conservation when the effective dielectric tensor is utilized with a dispersion relation which is formally the same as the dispersion relation for a homogeneous plasma. Along with the derivation of the effective dielectric tensor, we have stressed the somewhat subtle but fundamental differences between the effective tensor and other formulations which also incorporate inhomogeneity effects but do not satisfy the required symmetry. This emphasis is important since some confusion about these differences and the use of the effective dielectric tensor remains in the literature, as recently recognized by Istomin [24].

Regarding the subject of energy conservation, there are other approaches which do not utilize the BGI procedure and also claim energy conservation. Therefore we have to be careful about the precise meaning of the expression. When we say that the effective dielectric tensor ensures energy conservation, we mean that the absorption coefficient resulting from the dispersion relation really describes the energy exchange between a wave and particles, while other formalisms incorporate nonresonant terms which are related to variations of the wave amplitude due to variations of the group velocity in inhomogeneous media [14,17].

The expressions here derived for the components of the effective dielectric tensor are valid for any direction of propagation relative to the ambient magnetic field and to the inhomogeneity and are valid for the case of any azimuthally symmetric distribution function. The analysis has been restricted to the case of high frequency waves, so that the ions have been neglected in the dispersion relation. It was assumed that the ion distribution carries the current, which must exist in order to satisfy the equilibrium equations in the proposed inhomogeneous geometry.

The components for the effective dielectric tensor, given by Eq. (23), were shown to satisfy Onsager symmetry, which is required by the time invariance of the microscopical laws of motion, which rule the particle behavior, in contrast to other approaches found in the literature for the dielectric tensor of plasmas in inhomogeneous magnetic fields, which do not satisfy these fundamental requirements. We have discussed the derivation of these alternative approaches as compared to the derivation of the effective dielectric tensor of the present paper. In the derivation of these other approaches different approximations are utilized for the unperturbed orbits and the BGI corrections are not utilized. The comparison between these other approaches and the present formulation emphasizes the importance of a consistent derivation of the unperturbed orbits and the importance of the BGI correction in order to obtain a tensor that exhibits the formal symmetry features required for the correct description of the dielectric properties of the plasma and for the correct description of the wave-particle energy exchange.

Quantitative consequences of the use of the effective dielectric tensor have also been explored by the analysis of ordinary mode waves propagating along the direction of the inhomogeneity and perpendicularly to the direction of the magnetic field. The parameter measuring the degree of inhomogeneity has been assumed to be the aspect ratio of tokamaks, modeled by a slab of plasma. For this study we have assumed electrons described by a Maxwellian distribution and parameters ranging between those typical of small tokamaks and those expected for the next generation of large tokamaks. The outcome of the numerical analysis can be summarized as follows.

(i) There is a significant difference between the value of the absorption coefficient obtained with the use of the effective dielectric tensor in the dispersion relation and the value obtained with the use of other approaches utilized for comparison. Particularly, one of these approaches (approach A) predicts a null effect of the inhomogeneity for propagation parallel to the direction of inhomogeneity, while the present approach has shown a significant effect for a range of parameters. Another approach (approach B) has not incorporated relativistic effects and therefore has the inhomogeneity playing the role of resonance broadener alone. The absorption is predicted to be symmetric around the cyclotron frequency, which is in strong contrast to the outcome of the calculations made with the effective dielectric tensor. A third approach recently appearing in the literature includes relativistic effects, but does not take into account the BGI corrections. The resulting dielectric tensor also lacks Onsager symmetry and important qualitative and quantitative differences appear in the ensuing absorption coefficient, when compared to the results from the present formulation.

(ii) The inhomogeneity effect has been shown to decrease with the quantity $R^2B^2T^2$. For values of density, temperature, and magnetic field similar to those of JET or INTOR (International Tokamak Reactor), the absorption coefficient obtained for the ordinary mode has been quite close to the result predicted when inhomogeneous effects are neglected in the dispersion relation, even when a small aspect ratio is considered. However, for parameters typical of small tokamaks, the inhomogeneity effects can become meaningful when the aspect ratio decreases.

(iii) These local modifications in the absorption coefficient, even when meaningful, are almost canceled out when integrated along the trajectory. The integrated absorption, which is related to the optical depth of the plasma, has been shown to be relatively insensitive to the inhomogeneity effect.

(iv) The inhomogeneity in the magnetic field is most effective for frequencies near the electron cyclotron frequency, where the inhomogeneity effect may compete with relativistic effects, while in the case of inhomogeneities in the density or the temperature, the modification as compared to the homogeneous case is more noticeable in the wings of the absorption profile.

(v) For sufficiently small values of the quantity $R^2B^2T^2$, the modification in the absorption coefficient can be large enough to cause the appearance of a region of negative absorption near the electron cyclotron frequency, featuring a positive growth rate that is not due to reversed sign of the group velocity. This feature, which emerges due to the inhomogeneity, certainly deserves further investigation in order to shed light upon the detailed mechanism of the wave-particle interaction. In principle, one may say that, although the distribution function is Maxwellian, the existence of the nonequilibrium feature (the inhomogeneity in the magnetic field) is incorporated in the dielectric tensor. The resonance condition is modified by the inhomogeneity, which means that it is satisfied by a different population when compared with the homogeneous case. This modification in the resonant population can be regarded as an effective anisotropy, which has been shown to lead to negative absorption, for a given range of parameters.

These preliminary studies have been particularized for the case of the ordinary mode due to the relative simplicity of the dispersion relation. There are many other situations where interesting plasma phenomena occur in an inhomogeneous magnetic field, related to modes of propagation, ranges of frequency, or geometries other than those considered in the present paper. It is our intention to pursue our studies on the subject and report our findings in the near future.

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